Orders of Growth

When we talk about the efficiency of a function, we are often interested in the following: if the size of the input grows, how does the runtime of the function change? And what do we mean by "runtime"? Let’s look at the following examples first:

```python
def square(n):
    return n * n

def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n - 1)
```

- `square(1)` requires one primitive operation: * (multiplication). `square(100)` also requires one. No matter what input `n` we pass into `square`, it always takes one operation.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square(1)</td>
<td>1*1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>square(2)</td>
<td>2*2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>square(100)</td>
<td>100*100</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>square(n)</td>
<td>n*n</td>
<td>1</td>
</tr>
</tbody>
</table>
factorial(1) requires one multiplication, but factorial(100) requires 100 multiplications. As we increase the input size of n, the runtime (number of operations) increases linearly proportional to the input.

<table>
<thead>
<tr>
<th>input</th>
<th>function call</th>
<th>return value</th>
<th>number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>factorial(1)</td>
<td>1*1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>factorial(2)</td>
<td>2<em>1</em>1</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>factorial(100)</td>
<td>100<em>99</em>...<em>1</em>1</td>
<td>100</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>factorial(n)</td>
<td>n*(n-1)*...<em>1</em>1</td>
<td>n</td>
</tr>
</tbody>
</table>

Big-O notation is a way to denote an upper bound on the complexity of a function. For example, \( O(n^2) \) states that a function's run time will be no larger than the quadratic of the input.

- If a function requires \( n^3 + 3n^2 + 5n + 10 \) operations with a given input \( n \), then the runtime of this function is \( O(n^3) \). As \( n \) gets larger, the lower order terms (10, 5n, and \( 3n^2 \)) all become insignificant compared to \( n^3 \).
- If a function requires \( 5n \) operations with a given input \( n \), then the runtime of this function is \( O(n) \). The constant 5 only influences the runtime by a constant amount. In other words, the function still runs in linear time. Therefore, it doesn’t matter that we drop the constant.

### 1.1 Kinds of Growth

Here are some common orders of growth, ranked from no growth to fastest growth:

- \( O(1) \) — constant time takes the same amount of time regardless of input size
- \( O(\log n) \) — logarithmic time
- \( O(n) \) — linear time
- \( O(n^2), O(n^3), \text{ etc.} \) — polynomial time
- \( O(2^n) \) — exponential time (considered “intractable”; these are really, really horrible)

When using big-O notation, we always want to find the “tightest bound”. Recall that factorial(n) requires \( n \) multiplications. Its technically correct to say that factorial(n) is in \( O(n^2) \), since \( n^2 >= n \) for all values of positive values of \( n \), but its not very informative. Instead, we want to find the smallest big-O that factorial(n) belongs to. Since our implementation of factorial(n) must use at least \( n \) multiplications in all cases, we say its tightest bound is \( O(n) \).
1.2 Questions

1. What is the order of growth in time for the following functions? Use big-O notation.
   
   ```python
   def sum_of_factorial(n):
       if n == 0:
           return 1
       else:
           return factorial(n) + sum_of_factorial(n - 1)
   ```

2. ```python
   def fib_recursive(n):
       if n == 0 or n == 1:
           return n
       else:
           return fib_recursive(n - 1) + fib_recursive(n - 2)
   ```

3. ```python
   def fib_iter(n):
       prev, curr, i = 0, 1, 0
       while i < n:
           prev, curr = curr, prev + curr
           i += 1
       return prev
   ```

4. ```python
   def mod_7(n):
       if n % 7 == 0:
           return 0
       else:
           return 1 + mod_7(n - 1)
   ```

5. ```python
   def bonk(n):
       total = 0
       while n >= 2:
           total += n
           n = n / 2
       return total
   ```

6. ```python
   def bar(n):
       if n % 2 == 1:
           ```
```python
def foo(n):
    if n < 1:
        return
    if n % 2 == 0:
        return foo(n - 1) + foo(n - 2)
    else:
        return 1 + foo(n - 2)
```

What is the order of growth of \( \text{foo}(\text{bar}(n)) \)?

### 1.3 Extra Questions

1. Previously, we looked at the `is_prime` function. Here’s the code for it:

   ```python
def is_prime(n):
    if n == 1:
        return False
    k = 2
    while k < n:
        if n % k == 0:
            return False
        k += 1
    return True
```

What is the order of growth of `is_prime`?

How can we change `is_prime` so that it runs in \( O(\sqrt{n}) \)?

```python
def is_prime(n):
```
Consider a function that requires more than one recursive call. A simple example is a function that computes Fibonacci numbers:

```python
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

This type of recursion is called *tree recursion*, because it makes more than one recursive call in its recursive case. If we draw out the recursive calls, we see the recursive calls in the shape of an upside-down tree:

```
fib(4)
   / \
  fib(3) fib(2)
 / \   / \
fib(2) fib(1) fib(1) fib(0)
```

We could, in theory, use loops to write the same procedure. However, problems that are naturally solved using tree recursive procedures are generally difficult to write iteratively. As a general rule of thumb, whenever you need to try multiple possibilities at the same time, you should consider using tree recursion.
2.1 Questions

1. I want to go up a flight of stairs that has $n$ steps. I can either take 1 or 2 steps each time. How many different ways can I go up this flight of stairs? Write a function `count_stair_ways` that solves this problem for me. Assume $n$ is positive.

Before we start, what’s the base case for this question? What is the simplest input?

What does `count_stair_ways(n - 1)` represent? What does `count_stair_ways(n - 2)` represent?

Use those two recursive calls to write the recursive case:

```python
def count_stair_ways(n):
```
2. Here's a part of the Pascal's triangle:

<table>
<thead>
<tr>
<th>Item:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 0:</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 1:</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 2:</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 3:</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 4:</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Every number in Pascal’s triangle is defined as the sum of the item above it and the item that is directly to the upper left of it, use 0 if the entry is empty. Define the procedure `pascal(row, column)` which takes a row and a column, and finds the value at that position in the triangle.

```python
def pascal(row, column):
```

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3. The TAs want to print handouts for their students. However, for some unfathomable reason, both the printers are broken; the first printer only prints multiples of \( n_1 \), and the second printer only prints multiples of \( n_2 \). Help the TAs figure out whether or not it is possible to print an exact number of handouts!

First try to solve without a helper function. Also try to solve using a helper function and adding up to the sum.

```python
def has_sum(sum, n1, n2):
    '''
    >>> has_sum(1, 3, 5)
    False
    >>> has_sum(5, 3, 5) # 1(5) + 0(3) = 5
    True
    >>> has_sum(11, 3, 5) # 2(3) + 1(5) = 11
    True
    '''
```

2.2 Extra Questions

1. The next day, the printers break down even more! Each time they are used, Printer A prints a random \(x\) copies \(50 \leq x \leq 60\), and Printer B prints a random \(y\) copies \(130 \leq y \leq 140\). The TAs also relax their expectations: they are satisfied as long as they get at least \(\text{lower}\), but no more than \(\text{upper}\), copies printed. (More than \(\text{upper}\) copies is unacceptable because it wastes too much paper.)

Hint: Try using a helper function.

```python
def sum_range(lower, upper):
    #
    >>> sum_range(45, 60) # Printer A prints within this range
    True
    >>> sum_range(40, 55) # Printer A can print a number 56-60
    False
    >>> sum_range(170, 201) # Printer A + Printer B will print
    ... # somewhere between 180 and 200 copies total
    True
    #
```